

UNCONSTRAINED HAMILTONIAN FORMULATION OF GENERAL RELATIVITY WITH THERMO-ELASTIC SOURCES

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A new formulation of the Hamiltonian dynamics of the gravitational field interacting with (non-dissipative) thermo-elastic matter is discussed. It is based on a gauge condition which allows us to encode the six degrees of freedom of the “gravity + matter”-system (two gravitational and four thermo-mechanical ones), together with their conjugate momenta, in the Riemannian metric q_{ij} and its conjugate ADM momentum P^{ij} . These variables are not subject to constraints. We prove that the Hamiltonian of this system is equal to the total matter entropy. It generates uniquely the dynamics once expressed as a function of the canonical variables. Any function U obtained in this way must fulfil a system of three, first order, partial differential equations of the Hamilton-Jacobi type in the variables (q_{ij}, P^{ij}) . These equations are universal and do not depend upon the properties of the material: its equation of state enters only as a boundary condition. The well posedness of this problem is proved. Finally, we prove that for vanishing matter density, the value of U goes to infinity almost everywhere and remains bounded only on the vacuum constraints. Therefore the constrained, vacuum Hamiltonian (zero on constraints and infinity elsewhere) can be obtained as the limit of a “deep potential well” corresponding to non-vanishing matter. This unconstrained description of Hamiltonian General Relativity can be useful in numerical calculations as well as in the canonical approach to Quantum Gravity.

1. Introduction

General Relativity has been formulated as a Hamiltonian field theory by Arnowitt, Deser and Misner (1963, to be referred towards as ADM). The ADM paper contained not only the vacuum case but also a more general case of gravity interacting with Maxwell field and charged point particles. Since then, the canonical structure of General Relativity coupled with matter fields has been widely investigated in the case of perfect fluids and, more recently, in the case of multi-constituent fluids and superfluids (Comer & Langlois 1993, 1994).

It is well known that the gauge invariance of Einstein's theory implies that Hamiltonian General Relativity is a constrained theory. There have been many attempts to solve these constraints by imposing tricky gauge conditions, based e.g. on a certain "geometric time" used to parameterize space-time points. In the vacuum case, however, no satisfactory condition of this type has been found. In presence of matter it is easier to "gauge the time variable" in an invariant way (e. g. by using an extra scalar field, whose value is dynamically identified with time). Such approaches have an obvious drawback: not all Cauchy surfaces in spacetime are allowed in the hamiltonian description but only those which fulfil the gauge condition.

In the present paper we also use matter to gauge the time but – in contrast to the above approaches – *all the spacelike surfaces* are allowed as Cauchy data because our gauge condition does not fix the time variable but only its scale: it is fixed by the thermomechanical state of the matter.

For this purpose we need a formulation of continuum mechanics as a lagrangian field theory. As far as perfect fluids are concerned, two different field-theoretical approaches have been used: the Clebsh-potentials one which leads to the so called "non-canonical" Poisson structure (see e.g. Holm 1989), and an approach which, in contrast, may be called "canonical". In the latter approach, the dynamics is formulated in terms of three unconstrained field potentials, which assign to each spacetime point x a point $\xi(x)$ of an abstract, three dimensional "material space" \mathcal{B} , equipped with an appropriate geometric structure (Kijowski & Tulczyjew 1979, Künzle & Nester 1984). All the physical quantities describing the spacetime configuration of the fluid may be defined in terms of first derivatives of these three potentials and the equations of hydrodynamics may be formulated in terms of a system of 2-nd order, hyperbolic equations imposed on these potentials.

In a recent paper (Kijowski et al. 1990, to be referred towards as KSG) it was shown how to generalize the above approach to thermodynamically sensitive materials by adding a new, "material time" variable to the material space. This variable plays the role of a potential for the temperature, so that the resulting theory is described by four potentials. Such potentials can be re-parameterized in an arbitrary way, since a re-parameterization corresponds merely to a change of the "label" and the "clock" attached to each particle of the material. As a consequence, the theory of non-dissipative, isentropic fluids can be viewed as a "gauge type", lagrangian field theory.

Coupling fluid sources with gravity and passing to the Hamiltonian description one obtains a generalization of the ADM formalism in which the canonical variables are the Riemannian metric on each spacelike hypersurface and the matter fields, together with their conjugate momenta. The remarkable fact is that, in this approach, a gauge condition

can be used to obtain an unconstrained Hamiltonian description in which the matter degrees of freedom and those pertaining to gravity are both encoded in the metric and in its conjugate momentum only (KSG).

In the present paper we extend the above described results to the case of an arbitrary relativistic continuum (for instance a pre-stressed elastic solid) in the non-dissipative regime. The Hamiltonian description of the gravitational field obtained in this way displays interesting, universal geometric properties (recently, the same approach has been developed for the Hamiltonian dynamics of self-gravitating shells (Hajicek & Kijowski 1998)).

Besides its theoretical interest, an unconstrained formulation of General Relativity in matter can be useful in numerical approaches to the dynamics and the oscillations of strongly collapsed stars. Indeed, due to a process of crystallization of dense neutron matter, the crust of neutron stars probably exists in the form of a solid (see e.g. Shapiro & Teukolsky 1983, Haensel 1995). It is, moreover, worth mentioning that the main problem in the canonical approach to Quantum Gravity relies in the interpretation of the constraints in operatorial terms (see e.g. Kuchar 1993). As we shall see, our approach provides naturally a limiting procedure which allows us to treat the constrained, vacuum geometrodynamics as a limiting case of an unconstrained theory with “infinitesimally light” matter sources. Therefore, our formulation may prove to be useful also in this context.

The paper is organized as follows. In section 2 we give a thoroughly review of a “gauge-field-theoretical” formulation of relativistic continuum mechanics. This theory deserves, in our opinion, such a review because of its simplicity, internal beauty and universal properties. These properties are essential for our purposes.

In section 3 we extend the “gauge-type” description to the non-isothermal (non dissipative) case.

In section 4 we give the ADM formulation of the Einstein field equations in elastic media, and construct the corresponding variational principle in Hamiltonian form.

In section 5 we give the reduction of the theory with respect to the Gauss–Codazzi constraints. The main tool used is the *comoving gauge* description of the matter fields. This gauge consists in choosing the spacetime coordinates x^μ equal to the four “material spacetime” coordinates ξ^α , $\alpha = 0, 1, 2, 3$. The three conditions $x^a = \xi^a$ ($a = 1, 2, 3$), actually mean that we use comoving variables as space coordinates, whereas the condition $x^0 = \xi^0$ is used to define the time variable in terms of the temperature of the material. In this gauge, the six degrees of freedom of the composed “gravity + matter” system and their corresponding six momenta are completely described by the Riemannian metric q_{ij} of the Cauchy surface and by its conjugate ADM momentum P^{ij} . These data are not constrained: the Gauss–Codazzi equations play the role of implicit definition of the lapse function and the shift vector. We prove that the Hamiltonian of this system is equal to the total amount of entropy contained in the material under consideration. However, it must be expressed in terms of the canonical variables (q_{ij}, P^{ij}) . For this purpose the Gauss–Codazzi equations must be solved with respect to the lapse and the shift. In this way the entropy can be expressed as a function $S = S(X, Y_j, q^{ij})$, where X and Y_i denote the geometric objects built of q_{ij} and P^{ij} , which “stand on the left hand side” of the Gauss–Codazzi equations (the momentum enters into the Hamiltonian only *via* these objects).

The dynamics generated by S is unique. In particular, the lapse N and the shift N^k equal the derivatives of S with respect to X and Y_k respectively.

To obtain explicitly the Hamiltonian one needs to follow the above described algebraic procedure based on the “inversion” of the (former) constraints. There is, however, an underlying “differential structure” which, besides giving an equivalent way to calculate the Hamiltonian, reveals a rich mathematical content which is *universal* in the sense of being hidden in the Einstein–matter equations independently from the equation of state. Indeed, it turns out (section 6) that not all the functions of ten parameters $S = S(X, Y_j, q^{ij})$ can be obtained starting from all possible materials, described by all physically admissible state equations: we prove that the possible Hamiltonian have to fulfil a system of three first order, partial differential equations of the Hamilton–Jacobi type. These equations are universal in that they do not depend upon the specific matter taken into consideration, the matter properties being encoded only in the boundary value of S corresponding to the seven-dimensional subspace $\{Y_i = 0\}$. Since vanishing of Y_i implies vanishing of the shift, in this particular situation we are in the matter rest–frame and the seven parameters (X, q^{ij}) can be identified with the energy density and the strain tensor of the material, so that the function $S = S(X, 0, q^{ij})$ is the state equation. For a given material, the Hamiltonian $S = S(X, Y_j, q^{ij})$ can, therefore, be obtained by solving the Hamilton–Jacobi system on the 10-dimensional space of parameters (X, Y_j, q^{ij}) , with the state equation taken as boundary data. We prove that this boundary problem is well posed and may be solved uniquely by the method of characteristics.

The above description of the dynamics of the composed “gravity + matter” system sheds new light also on the constrained vacuum dynamics. Indeed, consider a family of functions $S_c(X, Y_j, q^{ij}) := S(X/c, Y_j/c, q^{ij})$ derived from a certain reference function $S = S_1$, describing a reference material, with c being a positive real number. We prove in Section 6 that the function S_c fulfils automatically the Hamilton–Jacobi equations if the reference function S_1 does, and that it describes a material whose energy (mass) is equal to the rescaled energy of the reference material (so that the new material is c times lighter - or heavier - than the reference one). In the limit $c \rightarrow 0$ the material becomes very light and its influence on the gravitational field can be neglected. On the other hand, the Hamiltonian S_c tends to infinity outside of the vacuum constraint submanifold $\{X = 0, Y_k = 0\}$, due to convexity properties of the entropy in physically reasonable cases. Therefore the vacuum Hamiltonian (zero on the constraints and infinity elsewhere) can be viewed as the limit of a sequence of non-constrained Hamiltonian forming a deep potential well with the constraint manifold taken as bottom.

Our Hamilton–Jacobi equations are derived in section 6 using a somewhat technical theorem. However, the physical origin of such equations is clear. Indeed, we prove in section 7 that they are equivalent to the *local* conservation of entropy, i.e. to the vanishing of the heat flow.

Finally, in section 8 we discuss in some detail the particular case of isotropic elastic sources. In this case the function of state does not depend of the entire strain tensor but only on its three invariants.

2. Relativistic mechanics of continua as a lagrangian field theory.

Relativistic hydrodynamics is a well established theory (see e.g. Anile & Choquet–Bruhat 1989). The relativistic description of elastic media is slightly less known. It has been formulated in many different, equivalent ways. The most important contributions are probably those due to DeWitt (1962), Souriau (1964), Hernandez (1970), Maugin (1971,1977,1978a), Carter & Quintana (1972), Glass & Winicour (1972), Carter (1973), Cattaneo (1973), Bressan (1978). For complete bibliography and comparative discussion of the various, equivalent formulations of the theory we refer the reader to Maugin (1978b) and Kijowski & Magli (1997).

In the present paper we use a “gauge-type” formulation of relativistic continuum mechanics, which can be used to describe *any* relativistic material (e. g. non-homogeneous, pre-stressed etc.) in a non-dissipative regime. This formulation can be considered as an obvious generalization of the “gauge-type” theory of relativistic elastic media (Kijowski & Magli 1992, 1997) and is essential for our purposes, since it is especially well adapted for the Hamiltonian description of “self-gravitating” continuum materials. The non-relativistic counterpart of this approach to continuum mechanics is known as the Piola (or inverse-motion) description (see e.g. Truesdell & Toupin 1960). For a complete formulation of finite elasticity in a language close to that of the present paper see Maugin’s (1993) book.

In this Section, the pseudo-Riemannian geometry $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$, signature $(-, +, +, +)$) of the general-relativistic spacetime \mathcal{M} is considered as given *a priori* (in Section 4 it will also become a dynamical quantity). To formulate the dynamical theory of a continuous material moving in \mathcal{M} , denote by \mathcal{B} the collection of all the idealized points (“molecules”) of the material, organized in an abstract 3-dimensional manifold, the *material space*. The spacetime configuration of the material is completely described by a mapping $\mathcal{G} : \mathcal{M} \rightarrow \mathcal{B}$, assigning to each spacetime point x the material point ξ (the specific “molecule”) which passes through this point. Each molecule $\xi \in \mathcal{B}$ follows, therefore, the spacetime trajectory defined as the inverse image $\mathcal{G}^{-1}(\xi) \subset \mathcal{M}$. Given a coordinate system (ξ^a) ($a = 1, 2, 3$) in \mathcal{B} and a coordinate system (x^μ) in \mathcal{M} , the configuration may, thus, be described by three fields $\xi^a = \xi^a(x^\mu)$ depending on four variables x^μ . We will show how to formulate the physical laws governing the mechanical properties of the material in terms of a system of second order, hyperbolic partial differential equations imposed on the fields. This way the mechanics of continua becomes a field theory and we may use its standard tools as variational principles, Noether theorem, Hamiltonian formulation with the underlying canonical (symplectic) structure of the phase space of Cauchy data etc.

As a first step we show that the kinematic quantities characterizing the spacetime configuration of the material, like the four-velocity u^μ , the matter current J^μ and the state of strain, can be encoded in the first derivatives of the fields. Consider the tangent mapping $\mathcal{G}_* : T_x \mathcal{M} \mapsto T_{\xi(x)} \mathcal{B}$, described by the (3×4) – matrix $(\xi^a_\mu) := (\partial_\mu \xi^a)$. We assume this matrix to have maximal rank and that its one-dimensional kernel to be time-like (in fact, the dynamical equations of the theory prevent the fields from violating these conditions in the future, once they are fulfilled by the Cauchy data). Vectors belonging to the kernel of (ξ^a_μ) are tangent to the world lines of the material, because the value of ξ^a remains constant on these lines. It follows that the velocity field u^μ can be defined as the unique future oriented vector field satisfying the conditions $u^\mu \xi^a_\mu = 0$ and $u^\mu u_\mu = -1$.

These four conditions allow to calculate u^μ uniquely in terms of the fields' derivatives and the metric (an explicit formula will be given below).

Given a spacetime configuration of the material, consider the push-forward of the contravariant physical metric $g^{\mu\nu}$ from the spacetime \mathcal{M} to the material space \mathcal{B} (Maugin 1978a):

$$G^{ab} := g^{\mu\nu} \xi^a_\mu \xi^b_\nu . \quad (1)$$

This tensor is obviously symmetric and positive definite. It defines, therefore, a (time-dependent) Riemannian metric in \mathcal{B} , carrying the information about the actual distances of adjacent particles of the material, measured in the local rest frame. Comparing this metric with an appropriate, pre-existing, geometric structure of \mathcal{B} , describing the mechanical structure of the material (like e. g. volume rigidity or shape rigidity) we can “decode” information about the local state of strain of the material at each instant of time: more the structure inherited from spacetime \mathcal{M} (*via* the tensor G) differs from the pre-existing structure of \mathcal{B} , higher is the state of strain of the material under consideration.

Below we give three different examples of such internal structures of \mathcal{B} , corresponding to fluids, isotropic elastic media and anisotropic (crystalline) materials, respectively. These structures *are not* dynamical objects of the theory: they are given *a priori* for any specific material. We stress, however, that the dynamical theory we are going to formulate in the sequel, is universal and applies to any material, whose physical properties may be described in terms of an appropriate geometric structure of \mathcal{B} .

Examples:

1. Volume structure

A 3-form (a scalar density)

$$\omega = r(\xi^a) d\xi^1 \wedge d\xi^2 \wedge d\xi^3 , \quad (2)$$

enables to measure the quantity of matter (number of particles or *moles*) contained in a volume $D \subset \mathcal{B}$ by integration over D . This “volume structure” is sufficient to describe the mechanical properties of a perfect fluid. Indeed, the ratio between the material's own volume form ω and the one inherited from spacetime *via* G^{ab} , i. e. the number

$$\rho := r \sqrt{\det G^{ab}} , \quad (3)$$

describes the actual density of the material (moles per cm^3), measured in the rest frame. Its inverse $v := 1/\rho$ is equal to the local, rest-frame specific volume of the fluid (cm^3 per mole). It contains the complete information about the state of strain of the fluid.

2. Metric structure

Elastic materials, displaying not only volume rigidity but also shape rigidity, are equipped with a Riemannian metric γ_{ab} , the *material metric*. It describes the “would be” rest-frame space distances between neighbouring “molecules”, measured in the locally relaxed state of the material. (To obtain such a locally relaxed state, we have to extract an “infinitesimal” portion from the bulk of the material. This way the influence of the rest of the material – possibly pre-stressed – is eliminated. Such an influence could otherwise make the relaxation impossible.)

The state of strain of the material is described by the “ratio” between the material metric and the physical metric inherited from \mathcal{M} *via* its actual spacetime configuration. This “ratio” may be measured by the tensor

$$S_a^b := \gamma_{ac} G^{cb} .$$

The material is locally relaxed at the point x if both structures coincide at x , i. e. if the actual, physical distances between material points in the vicinity of x agree with their material distances. This happens if and only if the strain tensor is equal to the identity tensor δ_a^b .

The simplest example of a material metric is obviously the flat, euclidean metric, corresponding to non-pre-stressed materials. A material carrying such a metric displays no “internal” or “frozen” stresses and can be embedded into flat Minkowski space *without generating any strain*. Such an embedding is impossible if the material metric has a non-vanishing curvature. Materials corresponding to curved metrics are, therefore, *pre-stressed* (in what follows, no specific assumption about γ_{ab} will be necessary).

Denoting by u_I the amount of internal energy (per mole of the material) of the elastic deformations, accumulated in an infinitesimal portion of the material during the deformation from the locally relaxed state to the actual state of strain. It is obvious that, for isotropic media, this function may depend on the deformation only *via* the invariants of the strain tensor. Since the metric γ carries automatically a volume structure $r := \sqrt{\det \gamma}$, we can take as one of these invariants the rest frame matter density ρ (or its inverse v), defined exactly as for fluids:

$$\rho = \sqrt{\det \gamma} \sqrt{\det G^{ab}} = \sqrt{\det S_a^b} . \quad (4)$$

As the remaining two invariants of S we can take e. g. its trace and the trace of its square:

$$h = S_a^a , \quad q = S_a^b S_b^a . \quad (5)$$

The physical meaning of these invariants is easily recognized if one considers a weak-strain limit (Hookean approximation). In this case the function u_I coincides with the standard formula of linear elasticity

$$u_I = \lambda(v) h^2 + 2\mu(v) q$$

where λ and μ are the Lamé coefficients, and h and q are the linear and the quadratic invariants of the strain, respectively.

3. Privileged deformation axis

For an anisotropic material (like a crystal) the energy of a deformation may depend upon its orientation with respect to a specific axis, reflecting the microscopic composition of the material. The information about the existence of such an axis may be encoded in a vector field E^a “frozen” in \mathcal{B} . We may, therefore, admit an additional dependence of the energy u_I upon the orientation of G with respect to one or several vectors E^a , i. e. upon the quantities $(G^{-1})_{ab} E^a E^b$.

To give an explicit formula for the velocity u^μ in terms of the fields ξ^a , consider the pull-back of the material volume form from the material space to the spacetime. This

pull-back is a differential 3-form in the 4-dimensional manifold \mathcal{M} , i. e. a vector density J which we call the *material current*. We have

$$J := \mathcal{G}^* \omega = r(\xi) d\xi^1(x) \wedge d\xi^2(x) \wedge d\xi^3(x) = r(\xi) \xi^1_{\nu} \xi^2_{\mu} \xi^3_{\sigma} dx^{\nu} \wedge dx^{\mu} \wedge dx^{\sigma} ,$$

On the other hand, every 3-form in \mathcal{M} may be written in the “vector-density” representation as

$$J = J^{\mu} (\partial_{\mu} \lrcorner dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3) . \quad (6)$$

This gives us the following formula for the components J^{μ} in terms of the fields ξ and their derivatives:

$$J^{\mu} = r(\xi) \epsilon^{\mu\nu\rho\sigma} \xi^1_{\nu} \xi^2_{\rho} \xi^3_{\sigma} , \quad (7)$$

(here we denote by $\epsilon^{\mu\nu\rho\sigma}$ the standard Levi-Civita tensor density). The vector density J is *a priori* conserved due to its geometric construction. Indeed, the exterior derivative of J is equal to the pull-back of the exterior derivative of ω , and the latter vanishes identically being a 4-form in the 3-dimensional space \mathcal{B} :

$$(\partial_{\mu} J^{\mu}) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = dJ = d(\mathcal{G}^* \omega) = \mathcal{G}^*(d\omega) = 0 , \quad (8)$$

or, equivalently:

$$\partial_{\mu} J^{\mu} = 0 .$$

Observe now that $J^{\mu} \xi^a_{\mu} \equiv 0$, since $\epsilon^{\mu\nu\rho\sigma} \xi^1_{\nu} \xi^2_{\rho} \xi^3_{\sigma} \xi^a_{\mu}$ is the determinant of a matrix with two identical columns. This means that J^{μ} is proportional to the velocity field and, therefore, may be written in the standard form:

$$J^{\mu} = \sqrt{-g} \rho u^{\mu} , \quad (9)$$

where the scalar $\rho = \sqrt{J^{\mu} J_{\mu} / g}$, with J^{μ} given by (7), is a nonlinear function of ξ^a_{μ} . Dividing J^{μ} given in (7) by “ $\sqrt{-g} \rho$ ” defined above, we obtain an explicit formula for u^{μ} in terms of ξ^a_{μ} .

Being a scalar, the quantity ρ can be calculated i. g. in the material rest frame, where $u^{\mu} = (1/\sqrt{-g_{00}}, 0, 0, 0)$. In this frame we have $\xi^a_0 = 0$ and, therefore,

$$\rho = \frac{J^0}{u^0 \sqrt{-g}} = \frac{r \det(\xi^a_k) \sqrt{-\det g^{\mu\nu}}}{\sqrt{-g_{00}}} = r \det(\xi^a_k) \sqrt{\det g^{kl}} = r \sqrt{\det G^{ab}} .$$

This proves that the quantity ρ defined this way coincides, indeed, with the rest frame matter density defined by (3).

In the present approach, the dynamical equations governing the evolution of the material under consideration can be derived from the lagrangian density $\Lambda := -\sqrt{-g} \epsilon = -\sqrt{-g} \rho e$, where $\epsilon = \rho e$ denotes the rest frame energy per unit volume of the material and e denotes the molar rest frame energy. The mechanical properties of each specific material are completely encoded in the function $e = e(G^{ab})$, which describes the dependence of its energy upon its state of strain. This function plays, therefore, the role of *equation of state* of the material. According to the general principles of relativity theory, it must contain

also the molar rest mass m , i.e. we have $e = m + u_I$. In generic situations, the equation of state depends also upon the point ξ *via* volume structure, metric structure, specific deformation axis or any other structure, which one may find necessary to describe the specific physical properties of the material. By abuse of notation we will, however, write $e = e(G^{ab})$ (instead of $e = e(\xi^a, G^{ab})$) whenever it does not lead to any misunderstanding. The density Λ is, therefore, a first order Lagrangian, depending upon the unknown fields ξ^a , their first derivatives ξ^a_{μ} (which enter through G^{ab}) and – possibly – the independent variables x^μ (which enter *via* the components $g_{\mu\nu}$ of the spacetime metric). The dynamical equations of the theory are, thus second order Euler–Lagrange equations and may be written as follows:

$$\partial_\mu p^\mu_a = \frac{\partial \Lambda}{\partial \xi^a}, \quad (10)$$

where we have introduced the momentum canonically conjugate to ξ^a :

$$p^\mu_a := \frac{\partial \Lambda}{\partial \xi^a_\mu}, \quad (11)$$

(for historical reasons we may call it the Piola–Kirchhoff momentum density).

The following identities may be immediately checked in the framework of the above theory (cfr. Kijowski & Magli 1992, 1997):

Proposition 1. (Belinfante – Rosenfeld identity)

The canonical energy-momentum tensor-density

$$-\mathcal{T}^\mu_\nu := p^\mu_a \xi^a_\nu - \delta^\mu_\nu \Lambda, \quad (12)$$

coincides with the symmetric energy-momentum tensor-density, i.e. the following identity holds:

$$\mathcal{T}_{\mu\nu} \equiv -2 \frac{\partial \Lambda}{\partial g^{\mu\nu}}.$$

(This identity is a straightforward consequence of the relativistic invariance of Λ . It may be checked explicitly by inspection if we take into account that both ξ^a_μ and $g_{\mu\nu}$ enter into Λ through their combination (1) only). \square

Proposition 2. (Noether identity)

$$-\nabla_\mu \mathcal{T}^\mu_\nu \equiv \left(\partial_\mu \frac{\partial \Lambda}{\partial \xi^a_\mu} - \frac{\partial \Lambda}{\partial \xi^a} \right) \xi^a_\nu. \quad (13)$$

Proof:

Differentiating (12) we obtain

$$-\partial_\mu \mathcal{T}^\mu_\nu = (\partial_\mu p^\mu_a) \xi^a_\nu + p^\mu_a \xi^a_{\nu\mu} - \partial_\nu \Lambda.$$

But

$$\partial_\nu \Lambda = \frac{\partial \Lambda}{\partial \xi^a} \xi^a_\nu + \frac{\partial \Lambda}{\partial \xi^a_\mu} \xi^a_{\nu\mu} + \frac{\partial \Lambda}{\partial g^{\sigma\kappa}} \frac{\partial g^{\sigma\kappa}}{\partial x^\nu}.$$

Taking into account the definition (11) of momenta, the symmetry of the second derivatives ($\xi^a_{\nu\mu} \equiv \xi^a_{\mu\nu}$) and the Belinfante – Rosenfeld identity, we obtain:

$$-\partial_\mu \mathcal{T}_\nu^\mu = \left(\partial_\mu \frac{\partial \Lambda}{\partial \xi^a_\mu} - \frac{\partial \Lambda}{\partial \xi^a} \right) \xi^a_\nu + \frac{1}{2} \mathcal{T}_{\sigma\kappa} \frac{\partial g^{\sigma\kappa}}{\partial x^\nu} .$$

Expressing the derivatives of the metric in terms of the connection coefficients we see that the last term gives exactly the contribution which is necessary to convert the partial derivative on the left hand side into the covariant derivative. This ends the proof. \square

We stress that the above identities are purely kinematical. They hold also for configurations which *do not* fulfil the dynamical equations. In particular, the Noether identity proves that the latter are actually *equivalent* to the energy-momentum conservation $\nabla_\mu \mathcal{T}_\nu^\mu = 0$. Indeed, the right hand side of (13) is automatically orthogonal to the matter velocity u^ν . This observation reduces the number of *independent* conservation laws from four down to three – exactly the number of the Euler-Lagrange equations.

To see, therefore, that the above theory describes correctly the laws of continuum mechanics, it is sufficient to calculate the energy-momentum tensor density \mathcal{T} and to identify it with the energy-momentum carried by the material under consideration. For this purpose we define, at each point of \mathcal{B} separately, the *response tensor* of the material

$$Z_{ab} := 2 \frac{\partial e}{\partial G^{ab}} , \quad (14)$$

or, equivalently

$$de(G^{ab}) = \frac{1}{2} Z_{ab} dG^{ab} . \quad (15)$$

As an example consider an isotropic elastic material, whose energy depends only upon the invariants (v, h, q) of the strain. Consequently, the response tensor may be fully characterized by the following response parameters

$$p = -\frac{\partial e}{\partial v} , \quad B = \frac{2}{v} \frac{\partial e}{\partial h} , \quad C = \frac{2}{v} \frac{\partial e}{\partial q} , \quad (16)$$

according to the formula

$$Z_{ab} = v \left(p (G^{-1})_{ab} + B \gamma_{ab} + C G_{ab} \right) .$$

The generating formula (15) reduces, in this case, to

$$de(v, h, q) = -p dv + \frac{1}{2} v B dh + \frac{1}{2} v C dq . \quad (17)$$

The response parameters defined above describe the reaction of the material to the strain. In particular, p describes the isotropic stress while B and C describe the anisotropic response as in the ordinary, non relativistic elasticity. The particular case of perfect fluid materials, corresponding to a constitutive function e which depends only on the specific

volume v , may be characterized by the vanishing of both the anisotropic responses, i.e. by equations $B = C \equiv 0$. Consequently, the response tensor for fluids is proportional to the physical metric $(G^{-1})_{ab}$ and the generating formula (15) reduces to the Pascal law: $de(v) = -p dv$.

For a general (not necessarily isotropic) material we have the following:

Proposition 3

The energy-momentum tensor-density of the above field theory is equal to:

$$\mathcal{T}_{\mu\nu} = \sqrt{-g} \, \rho (e \, u_\mu u_\nu + z_{\mu\nu}) , \quad (18)$$

where $z_{\mu\nu}$ is the pull-back of the response tensor Z_{ab} from \mathcal{B} to \mathcal{M} :

$$z_{\mu\nu} := Z_{ab} \xi_\mu^a \xi_\nu^b . \quad (19)$$

Proof:

We have:

$$\mathcal{T}_{\mu\nu} = 2 \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{-g} \rho e) . \quad (20)$$

But:

$$\begin{aligned} \frac{\partial \sqrt{-g}}{\partial g^{\mu\nu}} &= -\frac{1}{2} \sqrt{-g} g_{\mu\nu} , \\ \frac{\partial \rho}{\partial g^{\mu\nu}} &= \frac{1}{2\rho} \frac{\partial}{\partial g^{\mu\nu}} \left(\frac{J^\rho J^\sigma g_{\rho\sigma}}{g} \right) = \frac{1}{2} \rho (g_{\mu\nu} + u_\mu u_\nu) , \\ \frac{\partial e}{\partial g^{\mu\nu}} &= \frac{\partial e}{\partial G^{ab}} \frac{\partial G^{ab}}{\partial g^{\mu\nu}} = \frac{1}{2} Z_{ab} \frac{\partial (g^{\rho\sigma} \xi_\rho^a \xi_\sigma^b)}{\partial g^{\mu\nu}} = \frac{1}{2} z_{\mu\nu} . \end{aligned} \quad (21)$$

Inserting the above results in (20) we obtain eq. (18). \square

We recognize in formula (18) the standard energy momentum-tensor of continuum mechanics, composed of two components: the energy component “ $e \, u_\mu u_\nu$ ”, proportional to the velocity, and the *stress tensor* $\rho z_{\mu\nu}$ which is automatically orthogonal to the velocity. Equations (19) (or (14)) give the stress in terms of the strain (*stress – strain relations*) are uniquely implied by the constitutive equation $e = e(G^{ab})$ of the material.

The above formulation of continuum mechanics is, of course, invariant with respect to reparameterizations of the material space. Such reparameterizations may be interpreted as gauge transformations of the theory. They form the group of all the diffeomorphisms of \mathcal{B} . Physically, such transformations consist in changing merely the “labels” ξ^a assigned to the molecules of the material. Correspondingly, the fields ξ^a may be regarded as gauge potentials for the “elastic field strength” G^{ab} which is already gauge invariant. The three gauge potentials describe the three degrees of freedom of the material.

In Section 4 we are going to include also the gravitational field as a dynamical quantity. The group of gauge transformations of the entire theory (elasticity interacting with gravity) will be the product of the group of space-time diffeomorphisms (which is the gauge group of general relativity) by the group of diffeomorphisms of the material space.

3. Thermodynamics of isentropic flows.

It is relatively easy to extend the above theory to the thermodynamics of isentropic flows (no heat conductivity!). For this purpose we begin with the generating formula

$$de(G^{ab}, S) = \frac{1}{2} Z_{ab} dG^{ab} + TdS, \quad (22)$$

which generalizes (14) to the case of “thermodynamically sensitive” materials. Performing the Legendre transformation $TdS = d(TS) - SdT$, we obtain an equivalent formula with the Helmholtz free energy $f := e - TS$ playing the role of the generating function:

$$df(G^{ab}, T) = \frac{1}{2} Z_{ab} dG^{ab} - SdT. \quad (23)$$

In the particular case of perfect fluids the above formulae reduce to $de(v, S) = -pdv + TdS$ and to $df(v, T) = -pdv - SdT$, respectively.

Equation (23) suggest that the the temperature T may be interpreted as a strain and the entropy as the corresponding stress. This goes far beyond a formal analogy since it is possible to express the temperature in terms of derivatives of a new potential (ξ^0 , say), corresponding to a new, time-like dimension - the “material time” - in the material space. The configurations of the material turn out, therefore, to be described by *four* fields $\xi^\alpha = \xi^\alpha(x^\mu)$ ($\alpha = 0, 1, 2, 3$), and the “thermal strain T ” can be described in a way similar to that given by (1) for the elastic strain. The required formula for the temperature is the following (Kijowski & Tulczyjew 1982, KSG):

$$T = \beta u^\lambda \xi^0_{,\lambda}, \quad (24)$$

where β is a dimension-fixing constant.

Ansatz (24) can be viewed at a purely phenomenological level. Indeed, in the case of fluids the field theory derived from the Lagrangian $\Lambda = -\sqrt{-g} \rho f$ (where $f = f(v, T)$ stands for the molar free energy) describes correctly the relativistic hydrodynamics of isentropic flows (Kijowski & Tulczyjew 1982), and we shall show below that the same ansatz works for a generic elastic material as well. However, the potential ξ^0 has also a natural microscopic interpretation as the retardation of the proper time of the molecules with respect to the physical time calculated over averaged spacetime trajectories of the idealized continuum material. Indeed, consider the mean kinetic energy of the motion of the molecules of the material, calculated with respect to its rest frame. For temperatures not too high and average velocities \mathbf{v} much smaller than one (i. e. than the velocity of light) this energy equals $(1/2)m\mathbf{v}^2 = (3/2)\kappa T$ where κ is the Boltzmann constant. Consequently, the proper time τ of the particles is retarded with respect to the “physical” time x^0 (the affine parameter along the tangent to u^μ) according to formula:

$$\tau = \int \sqrt{1 - \mathbf{v}^2} dx^0 \approx \left(1 - \frac{3}{2} \frac{\kappa T}{m}\right) x^0.$$

It follows that, defining the retardation $\xi^0 = x^0 - \tau$, we have

$$T = \beta \frac{\partial \xi^0}{\partial x^0} ,$$

where $\beta = 2m/3\kappa$. Passing from the material rest frame to a general frame we get (24).

To show that the Lagrangian

$$\Lambda = -\sqrt{-g}\rho f(G^{ab}, T) .$$

describes correctly the thermo-mechanical behaviour of the material, observe that this is again a first order Lagrangian which depends upon the first derivatives ξ^α_μ of the potentials *via* the mechanical strains and the temperature. Now, we have four independent Euler-Lagrange equations corresponding to the variation with respect to four potentials:

$$\partial_\mu P^\mu_\alpha = \frac{\partial \Lambda}{\partial \xi^\alpha} , \quad (25)$$

where the momenta canonically conjugate to ξ^α (generalized Piola-Kirchhoff momenta) are defined as usual:

$$P^\mu_\alpha := \frac{\partial \Lambda}{\partial \xi^\alpha_\mu} .$$

Again, we have the following

Proposition 4

Both the Belinfante – Rosenfeld identity:

$$\mathcal{T}^\mu_\nu := - (P^\mu_\alpha \xi^\alpha_\nu - \delta^\mu_\nu \Lambda) \equiv -g^{\mu\sigma} \left(2 \frac{\partial \Lambda}{\partial g^{\sigma\nu}} \right) , \quad (26)$$

and the Noether identity:

$$-\nabla_\mu \mathcal{T}^\mu_\nu \equiv \left(\partial_\mu \frac{\partial \Lambda}{\partial \xi^\alpha_\mu} - \frac{\partial \Lambda}{\partial \xi^\alpha} \right) \xi^\alpha_\nu , \quad (27)$$

are valid for any field configuration of the above theory, not necessarily fulfilling the field equations. (The proof is an obvious generalization of the previous proofs.)

The Noether identity implies equivalence between the dynamical equations (25) of the theory and the energy-momentum conservation because the deformation gradient ξ^α_ν is a (4×4) -non-degenerate matrix. It is, however, worthwhile to notice that the variation with respect to ξ^0 produces simply the entropy conservation law. Indeed, due to (23) and (24) we have:

$$P^\mu_0 = -\sqrt{-g}\rho \frac{\partial f}{\partial T} \frac{\partial T}{\partial \xi^0_\mu} = \sqrt{-g}\beta S \rho u^\mu = \beta S J^\mu . \quad (28)$$

Because ξ^0 does not enter into the Lagrangian, the corresponding Euler-Lagrange equation reads

$$0 = \partial_\mu P^\mu_0 = \partial_\mu (\beta S J^\mu) = \beta J^\mu \partial_\mu S ,$$

which means that the amount of entropy contained in each portion of the material remains constant during the evolution.

The physical interpretation of the remaining three Euler-Lagrange equations is given by the following:

Proposition 5

The energy-momentum tensor of the above theory is given by the same formula (18), where the function e is defined by the Legendre transformation from (23) back to (22), i.e. by the formula $e := f + TS$.

Proof:

Due to the Rosenfeld–Belinfante identity we have

$$\mathcal{T}_\nu^\mu = 2 \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{-g} \rho f) .$$

Calculating the above derivative we obtain the same terms as in the proof of proposition 3 and, moreover, a term arising from the dependence of the Lagrangian upon T . Therefore, we have:

$$\mathcal{T}_\nu^\mu = \sqrt{-g} \rho (f u_\mu u_\nu + z_{\mu\nu}) + 2 \sqrt{-g} \rho \frac{\partial f}{\partial T} \frac{\partial T}{\partial g^{\mu\nu}} . \quad (29)$$

Now

$$\frac{\partial T}{\partial g^{\mu\nu}} = \frac{\partial}{\partial g^{\mu\nu}} \left(\frac{\beta J^\lambda \xi^0_\lambda}{\sqrt{-g} \rho} \right) = \beta J^\lambda \xi^0_\lambda \frac{\partial}{\partial g^{\mu\nu}} \frac{1}{\sqrt{-g} \rho} = -\frac{1}{2} T u_\mu u_\nu ,$$

where the first two formulae of (21) have been used. Inserting the above result in (29) and recalling that $\partial f / \partial T = -S$ we obtain formula (18) with $f + ST$ playing the role of e . \square

The above formulation of relativistic mechanics of continua as a lagrangian field theory leads in a natural way to its Hamiltonian counterpart. In the Hamiltonian formalism the infinite-dimensional phase space of Cauchy data for the fields on a given Cauchy surface $\{t = \text{const}\}$ is described by the four configurations variables ξ^α and their canonical conjugate momenta π_α , defined as derivatives of the Lagrangian with respect to the “velocities” $\dot{\xi}^\alpha = \xi^\alpha_0$:

$$\pi_\alpha := \frac{\partial \Lambda}{\partial \xi^\alpha_0} = P^0_\alpha . \quad (30)$$

The Poisson bracket between configurations and momenta assumes its canonical, delta-like form:

$$\{\pi_\beta(x), \xi^\alpha(y)\} = \delta^\alpha_\beta \delta(x, y) .$$

Assuming eqs. (30) to be invertible with respect to the $\dot{\xi}^\alpha$, the Hamiltonian $H_{el.}$ of the theory can be obtained by performing the Legendre transformation

$$H_{el.} := \pi_\alpha \dot{\xi}^\alpha - \Lambda = -\mathcal{T}_0^0 = -\sqrt{-g} T_0^0 ,$$

and therefore it is numerically equal to the energy density. Once expressed in terms of the canonical variables (ξ^α, π_β) and their spatial derivatives, $H_{el.}$ generates the Hamiltonian version of the field equations

$$\dot{f} = \{H_{el.}, f\}$$

4. ADM formulation of the Einstein field equations in elastic media.

In the present Section we are going to derive the canonical (Hamiltonian) formulation of General Relativity coupled to a thermo-elastic medium. First of all, we shall briefly review the corresponding ADM formulation for the vacuum case.

Given a “3 + 1 splitting” $\mathcal{M} = \Sigma \times \mathbf{R}^1$ of spacetime, describe the initial data on each initial value surface $\Sigma_t = \Sigma \times \{t\}$ by a 3-dimensional, Riemannian metric q_{ij} and the corresponding ADM momentum P^{ij} . The 4-dimensional spacetime metric is therefore equal to

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_i \\ N_j & q_{ij} \end{pmatrix},$$

where the quantities

$$N := \frac{1}{\sqrt{-g^{00}}},$$

$$N_i := g_{0i},$$

are the *lapse function* and the *shift vector*, respectively. The inverse metric reads

$$g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^j/N^2 & q^{ij} - N^i N^j/N^2 \end{pmatrix}. \quad (31)$$

The ADM momentum density is defined as

$$P^{ij} := \sqrt{\det q} (K q^{ij} - K^{ij})$$

where K_{ij} is the second fundamental form of Σ and K is its trace. The field equations split into a “non-dynamical” part (the four equations $\mathcal{G}^{00} = 0$ and $\mathcal{G}_k^0 = 0$) and a “dynamical” part ($\mathcal{G}_{ij} = 0$). The non-dynamical part gives four constraints for the Hamiltonian system described by (P^{ij}, q_{ij}) and the quantities N and N_k play the role of Lagrange multipliers. The constraints may be written as

$$X = 0,$$

$$Y_i = 0,$$

where we have defined the following objects:

$$X := \frac{1}{16\pi} \left[R - \frac{1}{q} \left(P^{ij} P_{ij} - \frac{1}{2} P^2 \right) \right],$$

$$Y_i := \frac{1}{8\pi\sqrt{q}} \nabla_k P_i^k. \quad (32)$$

In the above formulae, R and ∇ denote the Ricci scalar and the covariant derivative with respect to the 3-dimensional metric q_{ij} , respectively, P is the trace of P^{ij} and $q := \det q_{ij}$.

As in any constrained Hamiltonian system, the dynamics is not uniquely defined. In fact, one has the freedom of fixing the Lagrange multipliers N and N_k at each point of Σ and at each instant of “time” $t = x^0$. Such a freedom reflects the gauge invariance of General Relativity with respect to the group of spacetime diffeomorphisms.

The Hamiltonian equations governing this system can be shortly written as follows:

$$-\delta H = \frac{1}{16\pi} \int_{\Sigma} \dot{P}^{ij} \delta q_{ij} - \dot{q}_{ij} \delta P^{ij} . \quad (33)$$

This formula is the field-theoretical version of the standard, finite-dimensional Hamiltonian formula $-dH(p, q) = \dot{p}dq - qdp$. The gauge properties of the Hamiltonian formulation of General Relativity are reflected in the fact that the Hamiltonian vector field $(\dot{P}^{ij}, \dot{q}_{ij})$ is *not* uniquely given by the variation of the Hamiltonian. This happens because *not all* the variations $(\delta P^{ij}, \delta q_{ij})$ are allowed in (33) but only those respecting the constraints. Consequently, $(\dot{P}^{ij}, \dot{q}_{ij})$ are not given uniquely, but only up to vectors “orthogonal to the constraints” (in the sense of the symplectic structure $\int \delta P^{ij} \wedge \delta q_{ij}$).

In formula (33) we have skipped the usual volume term $(NX + N^i Y_i)$ because we are going to work “on shell”, where the constraints vanish *identically*. Consequently, the quantity H contains only “surface terms” (cfr. e.g. Misner et al. 1973). In the asymptotically flat case H equals the ADM-mass calculated at space infinity, while for compact Σ the Hamiltonian vanishes identically and the entire information about the dynamics may be retrieved from the constraints. For a discussion of a “quasi local” situation, where the mixed “initial value + boundary value” problem in a bounded subset $V \subset \Sigma$ with not trivial boundary ∂V is considered, we refer the reader to a recent paper (Kijowski 1997). It contains a general formula for the quasi-local Hamiltonian H expressed in terms of a surface integral over ∂V .

Coupling gravity to any matter theory consists in supplementing the above phase space of the gravitational Cauchy data by the Cauchy data for the matter fields. In the particular case of isentropic thermo-elasticity this means that the complete phase space will be described by twenty objects $(P^{ij}, q_{ij}, \pi_\alpha, \xi^\alpha)$. These objects have to fulfil to the constraints:

$$\begin{aligned} X &= \frac{N^2}{\sqrt{-g}} \mathcal{T}^{00} = \frac{N}{\sqrt{q}} \mathcal{T}^{00} \\ Y_i &= -\frac{N}{\sqrt{-g}} \mathcal{T}_i^0 = -\frac{1}{\sqrt{q}} \mathcal{T}_i^0 , \end{aligned} \quad (34)$$

where the matter energy density and the momentum density on the right hand side are given by (18). These quantities have to be expressed in terms of the canonical variables. This leads to an explicit form of the Gauss – Codazzi constraints, relating the geometric quantities X and Y_i with the material quantities (see equations (51) below).

The Hamiltonian formula generating the dynamics of the system now reads:

$$-\delta H = \frac{1}{16\pi} \int \dot{P}^{ij} \delta q_{ij} - \dot{q}_{ij} \delta P^{ij} + \int \dot{\pi}_\alpha \delta \xi^\alpha - \dot{\xi}^\alpha \delta \pi_\alpha , \quad (35)$$

and again it defines uniquely the dotted quantities *up to a gauge*, i. e. up to the symplectic annihilator of the constraints.

It was recently proved (Kijowski 1997) that the geometric quasilocal surface integral defining H is *universal* in the sense, that the Hamiltonian it defines is correct for any matter field and for the empty space, as well. One *should not*, however, conclude that

the dynamics of the gravitational field coupled to a matter field does not depend upon the specific properties of the matter. Indeed, for a given matter field, the Hamiltonian has to be considered as a function defined on the phase space of Cauchy data. These data must satisfy those *specific* constraints, which are implied by the *specific* properties of the considered material. There is no possibility to identify Cauchy data belonging to two different spaces, corresponding to different theories of matter. Hence, even if defined by the same boundary integral, the Hamiltonian corresponding to such two different matter fields generate two different field dynamics. In particular, H is always the ADM-mass in the asymptotically flat case and vanishes identically in the spatially-compact case.

5. Entropy picture and the reduction of the theory with respect to constraints.

Due to the diffeomorphisms invariance of the above described theory we are allowed to impose four conditions on the Cauchy data $(P^{ij}, q_{ij}, \pi_\alpha, \xi^\alpha)$ in order to reduce it with respect to the constraints. As far as the “spatial gauge” is concerned, it is somewhat natural to use the comoving frame, defined by the matter itself: $x^a = \xi^a$. This means that we identify the matter space \mathcal{B} with our Cauchy space Σ and that the velocity vector has only the time-component:

$$u^\mu = \frac{1}{\sqrt{-g_{00}}} \delta_0^\mu . \quad (36)$$

The main idea of the present approach consists in choosing a temporal gauge in which we identify also the physical time with the material time: $x^0 = \xi^0$. This 4-dimensional “comoving gauge” implies, therefore, that we have: $\xi^\alpha_\mu = \delta^\alpha_\mu$. Consequently, formula (36) implies that the gauge condition for the time variable x^0 is equivalent to

$$T = \beta \xi^0_\mu u^\mu = \frac{\beta}{\sqrt{-g_{00}}} . \quad (37)$$

Physically, the above equation means that the scale of time is no longer arbitrary but is uniquely fixed by the temperature of the material. We stress that, unlike many other gauge conditions used in General Relativity to fix the time variable (e. g. maximal surfaces, constant mean curvature etc.), this gauge condition does not impose any restriction on the choice of possible Cauchy surfaces. In particular, we will prove in Section 7 that the physical quantities describing the thermo-mechanical state of matter do not depend upon the particular choice of the Cauchy surface.

The gauge condition (37) generates an additional volume term in the Hamiltonian. This is due to the fact that, in this gauge, we have $\delta \xi^\alpha = 0$ and, consequently,

$$\dot{\pi}_\alpha \delta \xi^\alpha - \dot{\xi}^\alpha \delta \pi_\alpha = -\delta \pi_0 .$$

Being a complete variation, the above quantity may be carried to the left hand side of (35). But formulae (7) and (28) imply that

$$\pi_0 = P^0_0 = \beta S J^0 = \beta S r \epsilon^{0\nu\rho\sigma} \delta^1_\nu \delta^2_\rho \delta^3_\sigma = \beta S r .$$

Hence, the resulting Legendre transformation of (35) gives us the following generating formula:

$$-\delta\tilde{H} = \frac{1}{16\pi} \int \dot{P}^{ij} \delta q_{ij} - \dot{q}_{ij} \delta P^{ij} , \quad (38)$$

where the quantity

$$\tilde{H} := H - \beta \int Sr ,$$

plays the role of the total Hamiltonian of the system described by the canonically conjugate variables (P^{ij}, q_{ij}) only. It contains not only the surface term H but also a non-vanishing volume term proportional to the total entropy $\int Sr$ (molar entropy S integrated over the material space with respect to its volume structure r).

In the simplest case of a spatially compact spacetimes the quantity H vanishes and the dynamics is governed by the Hamiltonian

$$U := \beta r S , \quad (39)$$

which, due to eq. (38), generates the evolution equations

$$\begin{aligned} \dot{P}^{ij} &= 16\pi \frac{\delta U}{\delta q_{ij}} , \\ \dot{q}_{ij} &= -16\pi \frac{\delta U}{\delta P^{ij}} . \end{aligned} \quad (40)$$

In the case of a bounded piece of material V with non-vanishing boundary, the boundary term of the Hamiltonian provides us a tool to handle the behaviour of the canonical variables on ∂V , according to each specific boundary problem we want to consider. In fact, the evolution of the field within V (and the definition of the phase space of the system), is not complete unless we specify the appropriate boundary conditions for the fields (P^{ij}, q_{ij}) on ∂V . The dynamical equations of the theory are always given by (40), but they become closed only when a specific boundary value problem – and, consequently, a specific form of the boundary term H – is chosen. For a discussion of the boundary “phenomena” we refer to Kijowski (1997).

In order to be able to interpret the entropy as the Hamiltonian of the composed “gravity + matter” system we have first to interpret it as a thermodynamical generating function in the so called *entropy picture*. This picture is obtained from (22):

$$dS(e, G^{ab}) = \frac{1}{T} \left(de - \frac{1}{2} Z_{ab} dG^{ab} \right) , \quad (41)$$

where the constitutive equation $e = e(G^{ab}, S)$ has been solved with respect to the entropy and the latter has been taken as the generating function, that is:

$$S = S(e, G^{ab}) . \quad (42)$$

This function plays the role of the constitutive equation of the material in the entropy picture and defines *via* (41) the response of the material to changes of the control parameters

(e, G^{ab}) (in particular, $1/T$ plays role of the response to changes of e). In the particular case of perfect fluids, the entropy picture is defined by the well known formula

$$dS(e, v) = \frac{1}{T}(de - pdv) .$$

The function (42) becomes the Hamiltonian of the theory only once we are able to express its parameters in terms of the canonical variables data (P^{ij}, q_{ij}) . For this purpose we treat the four constraint equations (34) as implicit definitions of the lapse and the shift in terms of the four geometric quantities X and Y_i , together with the space metric q_{kl} . Once a specific material (i. e. a specific constitutive equation) has been chosen, all the thermo-elastic control parameters (e, G^{ab}) become uniquely defined as functions of the data (P^{ij}, q_{ij}) via the quantities X and Y_i (together with a possible *direct* dependence upon q_{kl}). Indeed, formula (31) proves that we have

$$G^{ij} = q^{ij} - V^i V^j , \quad (43)$$

where q^{ij} is the inverse space metric, whereas the “velocity” V is defined by the lapse and the shift as follows:

$$V^k := \frac{N^i}{N} .$$

On the other hand, formula (37) together with the geometric identity

$$g_{00} = N_k N^k - N^2 = N^2(V_k V^k - 1)$$

enables us to express uniquely the temperature in terms of the lapse and the shift. The problem consists, therefore, in solving the four constraints (e. g. in the form of equations (51) below) with respect to the four unknown quantities N, N_i . This is only possible for a specific material, when the constitutive equations (41) are explicitly given. For each chosen material these equations uniquely define the lapse and the shift in terms of the canonical variables. Finally, equation (43) and the constitutive equation enable us to express e and G_{ij} in terms of the latter. Inserting their values into (42) we finally obtain the entropy as a function (F , say) of the canonical variables:

$$F(X, Y_k, q^{kl}) = S(e(X, Y_k, q^{kl}), G^{ij}(X, Y_k, q^{kl})) . \quad (44)$$

This gives us the Hamiltonian U via formula (39).

The above procedure may be of little use in practice, because for realistic materials the resulting constraint equations may be highly non-linear and their analytic solution practically impossible to obtain. As will be explained in the next section, this difficulty can be circumvented and the Hamiltonian $F(X, Y_k, q^{kl})$ can be found as solution of a *universal* system of differential equations in 10 variables (X, Y_k, q^{kl}) . In this approach, the constitutive equations of a *specific* material enter only as boundary data on the surface $\{Y_k = 0\}$.

We stress that in the present picture the canonical variables (P^{ij}, q_{ij}) are *not constrained*. They carry the information about 6 independent degrees of freedom of the physical system under consideration: 2 for gravity and 4 for thermo-elasticity. Equations (34)

are no longer constraints: they allow us to reconstruct the lapse and the shift and, consequently, all the remaining physical quantities characterizing both the gravitational and the thermo-elastic fields, in terms of the canonical variables.

With respect to the vacuum case, the above theory consists in replacing the vanishing Hamiltonian $U \equiv 0$ on the constraint subspace $X = 0$ and $Y_i = 0$ by a non-trivial Hamiltonian $U = U(X, Y_i, q^{ij})$ and in relaxing completely the constraints. The dynamical equations generated this way for the quantities (P^{ij}, q_{ij}) carry not only the dynamics of the gravitational field, but also that of the matter coupled to gravity.

We are going to prove in the sequel that the theory of empty space may be obtained as a limiting case of theories with non-trivial matter, when the density of matter tends to zero. For this purpose let us consider a family of state equations

$$e_c(G^{ab}, S) = ce(G^{ab}, S) , \quad (45)$$

where c is a positive constant and $e = e(G^{ab}, S)$ corresponds to a reference material. The material described by the new state equation (45) differs from the reference material in the following way: the total mass of a piece of the *new material* is c times the mass of the same piece of the *reference material* (by *the same* piece we mean that it is in the same state of strain G^{ab} and contains the same amount of entropy). We will prove at the end of the next Section that the rescaled state equation (45) leads to the following Hamiltonian, when the material is coupled to gravity:

$$U_c(X, Y_i, q^{ij}) := U\left(\frac{X}{c}, \frac{Y_i}{c}, q^{ij}\right) . \quad (46)$$

The limit $c \rightarrow 0$ corresponds to a very light matter. In this regime the values of U_c become very big outside of the subspace $\{X = 0; Y_i = 0\}$ and remain bounded only on the constraints. This way the constraints arising in the vacuum case may be considered as a limiting case of a *very deep* “potential well”, corresponding to a *very light* matter.

6. Structure of the Hamiltonian.

Different materials are characterized by different Hamiltonian $U = \beta r S$. However, not all the functions of the ten parameters (X, Y_k, q^{kl}) may be obtained from an arbitrarily chosen constitutive function (42) of seven parameters through the Legendre transformation described above. Indeed, the function U has the following, universal properties:

Theorem 1

1) The function U fulfils the following system of three first-order partial differential equations:

$$2 \frac{\partial U}{\partial Y_l} \frac{\partial U}{\partial q^{kl}} = \left(\frac{\partial U}{\partial X} \right)^2 Y_k . \quad (47)$$

2) For vanishing Y_i , the shift vector vanishes and the function U satisfies the following boundary condition:

$$U(X, 0, 0, 0, q^{kl}) = \beta r S \left(\frac{X}{r \sqrt{\det q^{ij}}}, q^{kl} \right) , \quad (48)$$

where S is the constitutive function (42) of the material.

3) The above boundary value problem for equations (47) is well posed and may be solved by the characteristics method.

4) The lapse function N , and the shift vector N^k are uniquely given by the derivatives of U according to the following formulae:

$$\begin{aligned} N &= \frac{1}{\sqrt{q}} \frac{\partial U}{\partial X} , \\ N^l &= \frac{1}{\sqrt{q}} \frac{\partial U}{\partial Y_l} . \end{aligned} \quad (49)$$

Proof:

Due to equality (44) we have

$$\begin{aligned} \frac{1}{\beta r} \frac{\partial U}{\partial X} &= \frac{\partial S}{\partial e} \frac{\partial e}{\partial X} + \frac{\partial S}{\partial G^{ij}} \frac{\partial G^{ij}}{\partial X} , \\ \frac{1}{\beta r} \frac{\partial U}{\partial Y_k} &= \frac{\partial S}{\partial e} \frac{\partial e}{\partial Y_k} + \frac{\partial S}{\partial G^{ij}} \frac{\partial G^{ij}}{\partial Y_k} , \\ \frac{1}{\beta r} \frac{\partial U}{\partial q^{kl}} &= \frac{\partial S}{\partial e} \frac{\partial e}{\partial q^{kl}} + \frac{\partial S}{\partial G^{ij}} \frac{\partial G^{ij}}{\partial q^{kl}} . \end{aligned} \quad (50)$$

Now consider the constraints (34). Using formula (18) and recalling that $\xi_\mu^a = \delta_\mu^a$ in our gauge, these read:

$$\begin{aligned} X &= \frac{1}{v} \left(\frac{e}{1 - V^2} + Z_{kl} V^k V^l \right) , \\ Y_k &= -\frac{1}{v} \left(\frac{e}{1 - V^2} V_k + Z_{kl} V^l \right) , \end{aligned} \quad (51)$$

where $V^2 = q_{kl} V^k V^l$. These equations may be rewritten as

$$\begin{aligned} e &= v (X + Y_k V^k) , \\ Z_{kl} V^l &= - \left(v Y_k + \frac{e}{1 - V^2} V_k \right) . \end{aligned} \quad (52)$$

Due to formula (41), one has

$$\frac{\partial S}{\partial G^{kl}} = -\frac{1}{2} \frac{\partial S}{\partial e} Z_{kl} .$$

Using it and (43), we obtain

$$\begin{aligned} \frac{\partial S}{\partial G^{ij}} \frac{\partial G^{ij}}{\partial X} &= Z_{ij} V^j \frac{\partial V^i}{\partial X} \frac{\partial S}{\partial e} = - \left(v Y_i + \frac{e}{1 - V^2} V_i \right) \frac{\partial V^i}{\partial X} \frac{\partial S}{\partial e} , \\ \frac{\partial S}{\partial G^{ij}} \frac{\partial G^{ij}}{\partial Y_k} &= Z_{ij} V^j \frac{\partial V^i}{\partial Y_k} \frac{\partial S}{\partial e} = - \left(v Y_i + \frac{e}{1 - V^2} V_i \right) \frac{\partial V^i}{\partial Y_k} \frac{\partial S}{\partial e} , \\ \frac{\partial S}{\partial G^{ij}} \frac{\partial G^{ij}}{\partial q^{kl}} &= \left(\frac{\partial S}{\partial G^{kl}} + Z_{ij} V^j \frac{\partial V^i}{\partial q^{kl}} \right) \frac{\partial S}{\partial e} = \left[\frac{\partial S}{\partial G^{kl}} - \left(v Y_i + \frac{e}{1 - V^2} V_i \right) \frac{\partial V^i}{\partial q^{kl}} \right] \frac{\partial S}{\partial e} . \end{aligned}$$

Now we calculate the derivatives of the function $e = e(X, Y_k, q^{kl})$ using the first equation of (52) and the definition of v (see (3)):

$$v = \frac{1}{\rho} = \frac{1}{r \sqrt{\det (q^{kl} - V^k V^l)}} = \frac{1}{r \sqrt{\det q^{kl}} \sqrt{1 - V^2}} . \quad (53)$$

This way we obtain

$$\begin{aligned} \frac{\partial e}{\partial X} &= v + \left(v Y_i + \frac{e}{1 - V^2} V_i \right) \frac{\partial V^i}{\partial X} , \\ \frac{\partial e}{\partial Y_k} &= v V^k + \left(v Y_i + \frac{e}{1 - V^2} V_i \right) \frac{\partial V^i}{\partial Y_k} , \\ \frac{\partial e}{\partial q^{kl}} &= -\frac{e}{2} \left(q_{kl} + \frac{V_k V_l}{1 - V^2} \right) + \left(\frac{e V_j}{1 - V^2} + v Y_j \right) \frac{\partial V^j}{\partial q^{kl}} . \end{aligned}$$

Inserting the above results in eqs. (50) we obtain:

$$\begin{aligned} \frac{1}{\beta r} \frac{\partial U}{\partial X} &= v \frac{\partial S}{\partial e} , \\ \frac{1}{\beta r} \frac{\partial U}{\partial Y_k} &= v \frac{\partial S}{\partial e} V^k , \\ \frac{1}{\beta r} \frac{\partial U}{\partial q^{kl}} &= \frac{\partial S}{\partial G^{kl}} - \frac{1}{2} \frac{\partial S}{\partial e} e \left(q_{kl} + \frac{V_k V_l}{1 - V^2} \right) . \end{aligned} \quad (54)$$

Contracting the last equation with V^k and using the other two together with the vector constraint (52), we finally obtain (47).

To prove the validity of the initial condition (48) we observe that for $Y_k = 0$ the scalar constraint in (52) reduces to $e = vX$, whereas the vector constraint gives us

$$\left(\frac{e}{1 - V^2} q_{kl} + 2Z_{kl} \right) V^l = 0 .$$

For generic equations of state the tensor on the left is non-singular (it becomes singular only if one of the principal stresses - eigenvalues of the stress tensor - equals the large negative value $-\epsilon\rho/2(1 - V^2)$). Hence, in a generic situation V^l must vanish on initial data. Using (53) with $V = 0$ to express v in $e = vX$, we finally obtain (48).

To prove the integrability of the system composed by the three equations (47), we denote

$$P_X := \frac{\partial U}{\partial X} , P^i := \frac{\partial U}{\partial Y_i} , \Pi_{ij} := \frac{\partial U}{\partial q^{ij}}$$

and rewrite eqs. (47) as three Hamilton–Jacobi equations

$$\mathbf{H}_k (X, Y_i, q^{ij}, P_X, P^i, \Pi_{ij}) = 0 , \quad (55)$$

where the functions

$$\mathbf{H}_k := 2\Pi_{kl}P^l - P_X^2 Y_k ,$$

may be viewed as Hamiltonian defined on a 20-dimensional phase space \mathcal{P} parameterized by the coordinates Q_Σ, P^Σ , where $\Sigma = 1, \dots, 10$ and

$$Q_\Sigma = (X, Y_i, q^{ij}) .$$

On this phase space define the ordinary Poisson bracket as

$$\{\mathbf{F}, \mathbf{G}\} = \sum_{\Sigma=1}^{10} \left(\frac{\partial \mathbf{F}}{\partial Q_\Sigma} \frac{\partial \mathbf{G}}{\partial P^\Sigma} - \frac{\partial \mathbf{G}}{\partial Q_\Sigma} \frac{\partial \mathbf{F}}{\partial P^\Sigma} \right) .$$

Now, it is easy to check that

$$\{\mathbf{H}_k, \mathbf{H}_l\} = 0 .$$

This means that the three dynamical systems are in involution.

To prove point 3 of the theorem, we will propagate the initial value (48) of the function U over the characteristic lines of the three Hamiltonian. For this purpose we first calculate the values of the momenta P_X and Π_{ij} on the initial surface $\{Y_k = 0\}$ from the derivatives of the entropy (48) with respect to X and q^{ij} . Then we solve the equations (55) algebraically, with respect to the remaining three momenta P^i . This way we obtain, at each point of the initial surface, the complete set of initial data for the trajectories of the three Hamiltonian. The collection of all these data defines a 7-dimensional surface in the phase space \mathcal{P} . Because the Hamiltonian are in convolution, the trajectories starting from each point of the surface span a 3-dimensional characteristic subspace. The method of characteristics tells us that the function U must be constant on these subspaces (see e.g. Courant & Hilbert 1989). The collection of all the characteristic subspaces forms a 10-dimensional Lagrangian submanifold \mathcal{D} of \mathcal{P} and the function U is defined on \mathcal{D} . The solution of the problem is then obtained by projecting this function from \mathcal{D} down to the “configuration space” of the parameters Q_Σ .

Finally, to prove the last part of the Theorem, we observe that, due to (37) and (53), we have:

$$v \frac{\partial S}{\partial e} = \frac{v}{T} = \frac{v\sqrt{-g_{00}}}{\beta} = \frac{N\sqrt{q}}{\beta r}$$

and thus the first two equations of (54) reduce to (49). \square

Remark

The projection of the Lagrangian submanifold \mathcal{D} to the configuration space of the parameters Q_Σ may become singular on caustic surfaces. It was proved in KSG that, at least in the case of fluids, convexity of initial data (48), implied by the physical properties of the entropy function, excludes the existence of singularities and implies that the function U may be always constructed *globally*. In the case of a generic material this problem needs further investigations.

Finally, the following corollary of the previous Theorem shows how to reconstruct the vacuum gravity theory as a limiting case of the present theory, when matter becomes very light:

Corollary

If U is a solution of equations (47), the function U_c defined by formula (46) also satisfies the same equations and, therefore, may be taken as a possible Hamiltonian of the theory. It describes the dynamics of the material corresponding to the rescaled state equation (45).

Proof:

The first statement may be easily checked by inspection. Moreover, let us observe that the rescaling (45) of the energy is equivalent to the following rescaling in the entropy picture:

$$S_c(e, G^{ab}) = S\left(\frac{e}{c}, G^{ab}\right).$$

To prove that this relation is indeed satisfied by the material corresponding to the new Hamiltonian, consider the initial data (48) for U_c :

$$\begin{aligned} S_c(e, q^{kl}) &= \frac{1}{\beta r} U_c(X, 0, 0, 0, q^{kl}) = \frac{1}{\beta r} U\left(\frac{X}{c}, 0, 0, 0, q^{kl}\right) = \\ &= S\left(\frac{X}{cr\sqrt{\det q^{ij}}}, q^{kl}\right) = S\left(\frac{e}{c}, q^{kl}\right), \end{aligned}$$

which ends the proof. \square

7. Gauge invariance of the entropy.

Physically, the Hamilton–Jacobi conditions (47), imposed on the possible Hamiltonian U are equivalent to the invariance of the entropy with respect to spacetime diffeomorphisms. In fact, the function U “has to derive from its arguments” (X, Y_k, q^{kl}) the amount of the purely material quantity S . Performing a spacetime diffeomorphism we may change completely the data (X, Y_k, q^{kl}) . However, the value of U assigned to the new data must remain the same as before, since the amount of entropy contained in the material does not depend upon the parameterization of the initial data.

The invariance of the entropy with respect to purely 3-dimensional diffeomorphisms of \mathcal{B} (space diffeomorphisms) is automatically satisfied due to the fact that both the data (X, Y_k, q^{kl}) and the material structure defining the state equation are geometric objects defined on the matter space, whereas S is a scalar. Hence, only diffeomorphisms changing the time variable may be dangerous from this point of view. These “generalized boost transformations” correspond to non trivial changes of the Cauchy surface in the spacetime. We are going to prove that they also do not change the value of U .

In fact, consider a transformation which reduces to the identity on \mathcal{B} and consists in the translation of the material time

$$x^0 \longrightarrow x^0 + \varphi(x^k).$$

This transformation may be treated as generated by the Hamiltonian

$$U_\varphi(X, Y_k, q^{kl}) := \varphi(x^k) U(X, Y^k, q^{kl}).$$

Hence, we are going to prove that, for any function φ defined on \mathcal{B} , the dynamics generated by U_φ preserves the value of U_φ *at each point separately*, the global invariance being obvious because the integral of U_φ , i. e. the Hamiltonian, is always conserved by its own dynamics. In particular, for $\varphi \equiv 1$ we obtain the local entropy conservation with respect to the dynamics discussed previously.

Theorem 2

The equations (47) imply the local conservation of entropy with respect to the dynamics generated by U_φ .

Proof:

To simplify the proof it is convenient to introduce the vector density associated to the vector Y_i :

$$y_i := \sqrt{q} Y_i = \frac{1}{8\pi} \nabla_k P_i^k, \quad (56)$$

and to express the Hamiltonian U_φ in terms of this variable and the *covariant* (instead of the contravariant) metric. We denote:

$$W_\varphi(X, y_k, q_{kl}) := \varphi(x^l) U \left(X, \frac{y_k}{\sqrt{q}}, q^{kl} \right).$$

First of all, we are going to derive the evolution equations (40) with the Hamiltonian U replaced by W_φ . For this purpose we calculate the total variation δW_φ :

$$\delta W_\varphi = \frac{\partial W_\varphi}{\partial X} \delta X + \frac{\partial W_\varphi}{\partial y_k} \delta y_k + \frac{\partial W_\varphi}{\partial q_{kl}} \delta q_{kl}. \quad (57)$$

Using the definition (32), we obtain

$$16\pi\delta X = \delta R + \frac{1}{q} \left(P^{ij} P_{ij} - \frac{P^2}{2} \right) q^{kl} \delta q_{kl} - \frac{1}{q} \left[(2P_{kl} - P q_{kl}) \delta P^{kl} + (2P_m^k P^{ml} - P P^{kl}) \delta q_{kl} \right].$$

The variation of the Ricci scalar may be written as follows:

$$\delta R = -R^{kl} \delta q_{kl} + (\delta_s^l q^{mn} - \delta_s^n q^{ml}) \Gamma_{lk}^k \delta \Gamma_{mn}^s + \partial_l [(\delta_s^l q^{mn} - \delta_s^n q^{ml}) \delta \Gamma_{mn}^s],$$

therefore we obtain:

$$\begin{aligned} 16\pi\delta X = & \left[-R^{kl} + \frac{1}{q} \left(P^{ij} P_{ij} - \frac{P^2}{2} \right) q^{kl} - \frac{1}{q} (2P_m^k P^{ml} - P P^{kl}) \right] \delta q_{kl} + \\ & - \frac{1}{q} (2P_{kl} - P q_{kl}) \delta P^{kl} + (\delta_s^l q^{mn} - \delta_s^n q^{ml}) \Gamma_{lk}^k \delta \Gamma_{mn}^s + \partial_l [(\delta_s^l q^{mn} - \delta_s^n q^{ml}) \delta \Gamma_{mn}^s]. \end{aligned} \quad (58)$$

Due to (56) we have:

$$8\pi\delta y_k = \partial_l (P^{lm} \delta q_{km} + q_{km} \delta P^{lm}) - \frac{1}{2} (P^{lm} \delta q_{lm,k} + q_{lm,k} \delta P^{lm}). \quad (59)$$

Inserting formulae (58) and (59) in (57), collecting the terms containing independent variations and eliminating all the boundary terms (total divergences) we finally obtain the following evolution equations generated by W_φ :

$$\begin{aligned} \dot{q}_{kl} &= \frac{2}{q} \frac{\partial W_\varphi}{\partial X} \left(P_{kl} - \frac{1}{2} P q_{kl} \right) + q_{km} \nabla_l \frac{\partial W_\varphi}{\partial y_m} + q_{lm} \nabla_k \frac{\partial W_\varphi}{\partial y_m} , \\ \dot{P}^{kl} &= \frac{\partial W_\varphi}{\partial X} \left[-R^{kl} + \frac{1}{q} \left(P^{ij} P_{ij} - \frac{P^2}{2} \right) q^{kl} - \frac{1}{q} (2P_m^k P^{ml} - P P^{kl}) \right] + \\ &\quad + \nabla^k \nabla^l \frac{\partial W_\varphi}{\partial X} - q^{kl} \nabla_m \nabla^m \frac{\partial W_\varphi}{\partial X} - P^{km} \nabla_m \frac{\partial W_\varphi}{\partial y_l} - P^{lm} \nabla_m \frac{\partial W_\varphi}{\partial y_k} + \\ &\quad + \nabla_m \left(P^{kl} \frac{\partial W_\varphi}{\partial y_m} \right) + 16\pi \frac{\partial W_\varphi}{\partial q_{kl}} , \end{aligned} \quad (60)$$

(the above formulae may be rewritten in a somewhat more familiar form if we replace the derivatives of W_φ with respect to X and y_l by the lapse and the shift, using eqs. (49)). To calculate \dot{W}_φ , we may simply rewrite formulae (58) and (59), replacing the variations of q_{kl} and P^{kl} by their time derivatives. This way we obtain:

$$\begin{aligned} 16\pi \dot{X} &= \left[-R^{kl} + \frac{1}{q} \left(P^{ij} P_{ij} - \frac{P^2}{2} \right) q^{kl} - \frac{1}{q} (2P_m^k P^{ml} - P P^{kl}) \right] \dot{q}_{kl} + \\ &\quad - \frac{1}{q} (2P_{kl} - P q_{kl}) \dot{P}^{kl} + \nabla_l (\nabla_m \dot{q}^{ml} - \nabla^l \dot{q}_m^m) , \\ 8\pi \dot{y}_k &= \nabla_l (P^{lm} \dot{q}_{km} + q_{km} \dot{P}^{lm}) - \frac{1}{2} P^{lm} \nabla_k q_{lm} . \end{aligned} \quad (61)$$

Finally, inserting (60) into (61) we obtain:

$$\begin{aligned} \dot{X} &= \frac{\partial W_\varphi}{\partial y_l} \nabla_l X + \frac{2}{q} y^l \nabla_l \frac{\partial W_\varphi}{\partial X} + \frac{1}{q} \frac{\partial W_\varphi}{\partial X} \nabla_l y^l - \frac{2}{q} \left(P_{kl} - \frac{1}{2} P q_{kl} \right) \frac{\partial W_\varphi}{\partial q_{kl}} , \\ \dot{y}_k &= -\frac{\partial W_\varphi}{\partial X} \nabla_k X + \nabla_l \left(y_k \frac{\partial W_\varphi}{\partial y_l} \right) + y_l \nabla_k \frac{\partial W_\varphi}{\partial y_l} + 2 \nabla_l \left(q_{km} \frac{\partial W_\varphi}{\partial q_{lm}} \right) . \end{aligned}$$

Plugging the above results into formula:

$$\dot{W}_\varphi = \frac{\partial W_\varphi}{\partial X} \dot{X} + \frac{\partial W_\varphi}{\partial y_k} \dot{y}_k + \frac{\partial W_\varphi}{\partial q_{kl}} \dot{q}_{kl} ,$$

one may readily check the following result:

$$\dot{W}_\varphi = \nabla_l \left[y_k \frac{\partial W_\varphi}{\partial y_k} \frac{\partial W_\varphi}{\partial y_l} + \frac{1}{q} \left(\frac{\partial W_\varphi}{\partial X} \right)^2 q^{lk} y_k + 2 \frac{\partial W_\varphi}{\partial y_k} q_{km} \frac{\partial W_\varphi}{\partial q_{ml}} \right] . \quad (62)$$

Now, we come back to our variables Y_i and to the function U . For this purpose we observe that

$$\begin{aligned} \frac{\partial W_\varphi}{\partial X} &= \varphi \frac{\partial U}{\partial X} , \\ \frac{\partial W_\varphi}{\partial y_k} &= \frac{1}{\sqrt{q}} \varphi \frac{\partial U}{\partial Y_i} , \\ \frac{\partial W_\varphi}{\partial q_{kl}} &= \varphi \left(-q^{ik} q^{jl} \frac{\partial U}{\partial q^{ij}} + \frac{1}{2} Y_i \frac{\partial U}{\partial Y_i} \right) . \end{aligned}$$

Plugging these into formula (62) we finally obtain

$$\varphi \dot{U} = \partial_l Q^l ,$$

where we have defined

$$Q^l := q^{lm} \frac{\varphi^2}{\sqrt{q}} \left[\left(\frac{\partial U}{\partial X} \right)^2 Y_m - 2 \frac{\partial U}{\partial Y_k} \frac{\partial U}{\partial q^{km}} \right] . \quad (63)$$

This vector density vanishes due to the Hamilton–Jacobi equations (47). This ends the proof. \square

Physically, the vector density Q^l is equal to the entropy current (i. e. to the heat flow). We conclude that the Hamilton–Jacobi conditions imposed on U are a consequence of the fact that we are considering only isentropic phenomena, for which the heat flow vanishes identically.

8. Isotropic elastic media.

In the case of an isotropic material, the function of state depends, besides of e , only on the three control parameters (v, h, q) (cfr. section 2, example 2). Such parameters contain also the material metric which is *a priori* given for each material. Hence, the solution depends also upon γ_{ij} , but the Hamiltonian must be invariant with respect to local (i. e. at each point independently) isometries of γ . This implies that S may depend, besides of the scalar X , only upon the invariants of the matrix $\chi^i_j := q^{ik} \gamma_{kj}$ and upon invariants built of the vector Y_i . We choose the following set for the invariants of χ :

$$\begin{aligned} \mathcal{Z} &:= \det \chi , \\ \mathcal{H} &:= \text{Tr} \chi , \\ \mathcal{L} &:= \text{Tr} \chi^2 - (\text{Tr} \chi)^2 \end{aligned} \quad (64)$$

For the remaining invariants we choose the lengths of Y_i calculated with respect to three different metric tensors:

$$\begin{aligned} \mathbf{r} &:= \gamma^{ij} Y_i Y_j , \\ \mathbf{s} &:= q^{ij} Y_i Y_j , \\ \mathbf{t} &:= q^{ij} q^{kl} \gamma_{jl} Y_i Y_j . \end{aligned} \quad (65)$$

It is, therefore, obvious that the function S will depend upon X , Y_i and q^{kl} via seven invariants only

$$S = F(X, \mathcal{Z}, \mathcal{H}, \mathcal{L}, \mathbf{r}, \mathbf{s}, \mathbf{t}) . \quad (66)$$

The equations (47) may be rewritten in terms of the above invariants in the following way:

$$\begin{aligned} p_Z p_r + 2p_{\mathcal{L}} p_s + p_{\mathcal{H}} p_t &= 0 , \\ p_t (r p_r + s p_s + t p_t) &= p_t (\mathcal{Z} p_Z + \mathcal{H} p_{\mathcal{H}} + \mathcal{L} p_{\mathcal{L}}) + 2p_{\mathcal{L}} p_r + p_s p_{\mathcal{H}} \\ \frac{1}{2} (p_X)^2 - 2p_s (r p_r + s p_s + t p_t) &= 2(p_{\mathcal{H}} - 2\mathcal{H} p_{\mathcal{L}}) p_r + (\mathcal{L} p_{\mathcal{H}} + 4\mathcal{Z} p_{\mathcal{L}}) p_t + \\ &+ 2(\mathcal{L} p_{\mathcal{L}} + \mathcal{Z} p_Z) p_s + \frac{1}{3} p_t \{ 2s p_r + 2t p_s + [2\mathcal{H} t + 2\mathcal{Z} r + \mathcal{L} s] p_t \} , \end{aligned} \quad (67)$$

where by p with a subscript we denote the derivative of F with respect to the corresponding variable, e.g. $p_X = (\partial F / \partial X)$.

For vanishing Y_i , i. e. for $r = s = t = 0$, the shift vector vanishes and, therefore, the tensor χ coincides with the strain h . Consequently, its invariants $(\mathcal{Z}, \mathcal{H}, \mathcal{L})$ may be calculated in terms of the invariants (v, h, q) of the strain. Moreover, the Hamiltonian constraint still gives $X = e/v$. This implies that the function F satisfies the following boundary condition:

$$F(X, \mathcal{Z}, \mathcal{H}, \mathcal{L}, 0, 0, 0) = S(X/\sqrt{\mathcal{Z}}, 1/\sqrt{\mathcal{Z}}, \mathcal{H}, \mathcal{L} + \mathcal{H}^2) , \quad (68)$$

where S is the constitutive function of the material.

Finally, we are going to discuss two simple examples, which correspond to particularly simple choice of the equation of state:

1. Perfect fluids

For a perfect fluid the equation of state depends, besides of e , only on the specific volume. Therefore $p_{\mathcal{H}} = p_{\mathcal{L}} = 0$ identically. Then the first equation in (67) gives $p_r = 0$ while the second one then implies $p_t(sp_s + tp_t) = p_t \mathcal{Z} p_{\mathcal{Z}}$. Choosing the solution $p_t = 0$ we end up with a function $F = F(X, \mathcal{Z}, s)$ which has to satisfy the partial differential equation

$$\left(\frac{\partial F}{\partial X} \right)^2 - 4s \left(\frac{\partial F}{\partial s} \right)^2 = 4\mathcal{Z} \frac{\partial F}{\partial \mathcal{Z}} \frac{\partial F}{\partial s} .$$

The above equation simplifies considerably if put

$$\mathcal{Y} = \frac{s}{\mathcal{Z}} ,$$

in fact in such a case we have (KSG):

$$\left(\frac{\partial F}{\partial X} \right)^2 - 4 \frac{\partial F}{\partial \mathcal{Y}} \frac{\partial F}{\partial \mathcal{Z}} = 0 . \quad (69)$$

2. An elastic material resembling a perfect fluid

Consider an elastic material for which the state equation, besides of e , depends only on the trace of the strain tensor. Physically, this material has always the same response to all the strains that change its volume without changing its shape. Therefore, it may be considered as a “counterpart” of the perfect fluid (the response of a perfect fluid to strains that do change its shape without changing its volume vanishes). If F depends on \mathcal{H} only, we have $p_{\mathcal{Z}} = p_{\mathcal{L}} = 0$ and the first equation in (67) gives $p_t = 0$. The second one then implies $p_s = 0$. Therefore we end up with a function $F = F(X, \mathcal{H}, r)$ which has to satisfy the partial differential equation

$$\left(\frac{\partial F}{\partial X} \right)^2 - 4 \frac{\partial F}{\partial r} \frac{\partial F}{\partial \mathcal{H}} = 0 .$$

The above equation is formally identical with the perfect-fluid equation (69).

For a more detailed discussion of the isotropic case, see Iacoviello et al. (1996).

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